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ABSTRACT

The Hugoniot relations, which express conservation of mass, momentum, and energy across a discontinuity, are derived. The relations are applied to a perfect gas for illustration, and graphical methods of solving shock problems are presented.

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INTRODUCTION

This article is divided into three sections that are not necessarily dependent upon one another. The first is the derivation of the Hugoniot relations which express conservation of mass, momentum, and energy across a discontinuity. The second is the application of these relations to a perfect gas, and the third is the solution of shock problems by graphical methods.

The source of this material is to be found in Hugoniot's original paper in the Journal de L'Ecole Polytechnique, 1889. The method of solving shock problems graphically was first shown to me by Alan Kaufman.

In this analysis a shock is a compression wave with an infinite pressure gradient. The shock width is small, so a shock wave can be replaced by a surface across which the pressure, density, material velocity, and internal energy change discontinuously.

The equations of state used are in the form of pressure as a function of volume and internal energy. It is convenient to use relative volume units, or the ratio of the true volume to its original volume. An equivalent ratio is the ratio of the original density to the present density.

I. DERIVATION OF HUGONIOT RELATIONS

Given a fluid in an initial state E_0, ρ_0, P_0, and U_0 representing energy, density, pressure, and material velocity, respectively. Consider a shock with velocity S (with respect to the gas velocity in front) starting from the end and traveling through the fluid, changing the state from E_0, ρ_0, P_0, U_0 to E_1, ρ_1, P_1, U_1. In a time $\underline{t}$ a length $L = St$ will have been swept out (Fig. 1a). The velocity of the rear surface relative to the fluid is $(U_1 - U_0)$; therefore the rear surface will have been displaced $(U_1 - U_0)t$ during the time the shock has traveled the length $L = St$ (Fig. 1b).

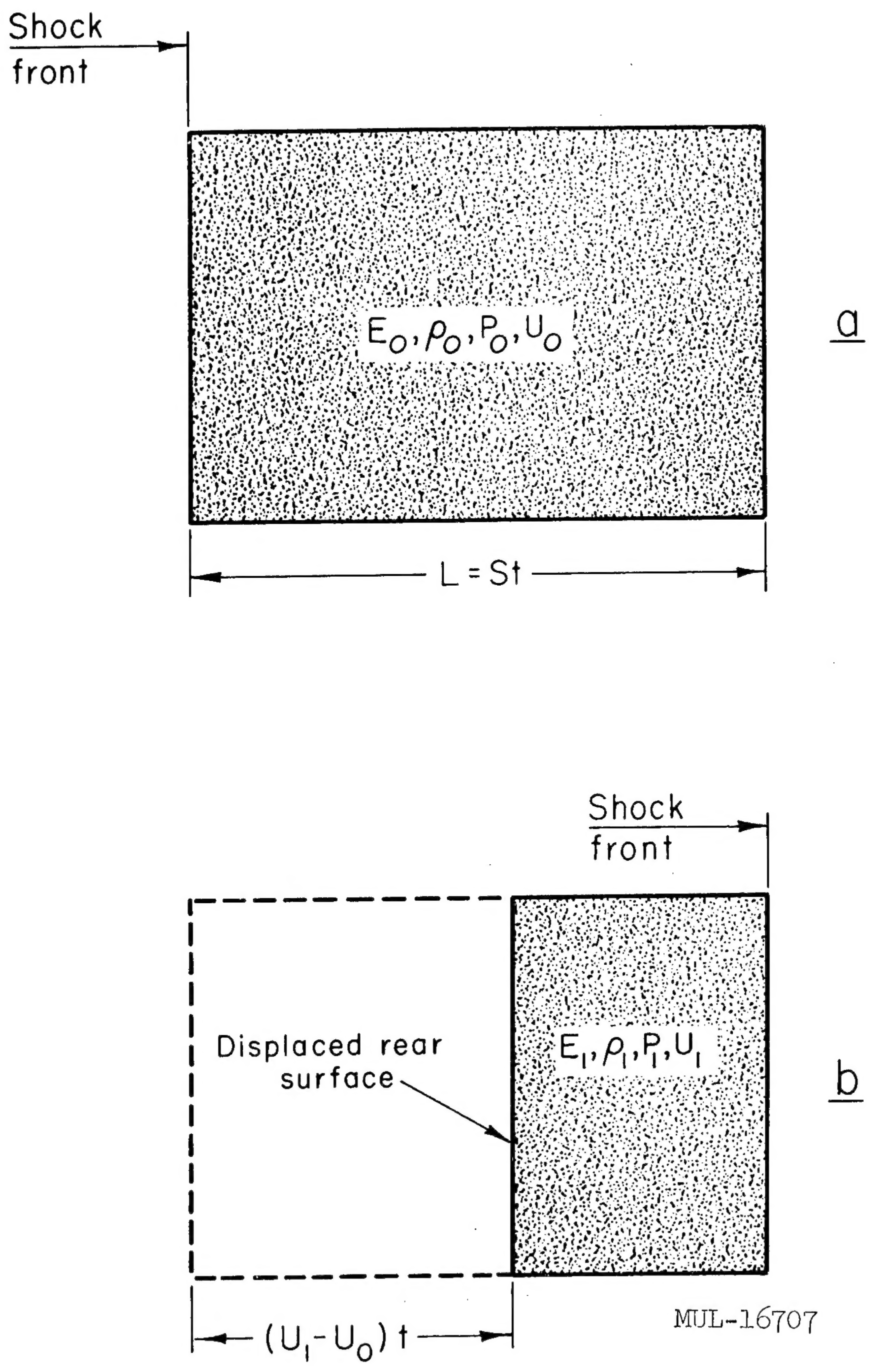

Fig. 1. Portion of fluid of length L in (a) initial state, just as shock front strikes, and (b) final state, just after shock front has swept through.

1. Conservation of Mass

For the length of material being considered, conservation of mass requires the mass before and after the shock has passed to be the same. The cross section is considered to be unity.

$$St\rho_0 = [St - (U_1 - U_0)t]\,\rho_1,$$

$$S\rho_0 = S\rho_1 - (U_1 - U_0)\rho_1,$$

$$S(\rho_0 - \rho_1) = -(U_1 - U_0)\rho_1,$$

$$S = \frac{(U_1 - U_0)\rho_1}{\rho_1 - \rho_0} = \frac{(U_1 - U_0)V_0}{V_0 - V_1} \qquad \text{(I.1)}$$

where*

$$V_1 = \frac{\rho_0}{\rho_1},$$

$$V_0 = \frac{\rho_0}{\rho_0}.$$

2. Conservation of Momentum

Conservation of momentum for the length L requires that the net force × time equal the change in momentum.

$$(P_1 - P_0)t = \rho_0 LU_1 - \rho_0 LU_0$$

$$= \rho_0 StU_1 - \rho_0 StU_0,$$

$$P_1 - P_0 = \rho_0 S(U_1 - U_0). \qquad \text{(I.2)}$$

Substituting $(U_1 - U_0)$ from equation (I.1) we get

$$S^2 = \frac{V_0}{\rho_0}\left(\frac{P_1 - P_0}{V_0 - V_1}\right). \qquad \text{(I.2a)}$$

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*Here the volumes are referred to the density ρ_0, making the relative volume V_0 equal to 1. V_0 is carried through the equations, even though it is 1, to describe the general case where the volumes are referred to a reference density (ρ_{ref}) that is not the density ρ_0 ahead of the shock. In this case we would have $V_0 = \rho_{ref}/\rho_0$ and $V_1 = \rho_{ref}/\rho_1$.

Using equations (I.1) and (I.2) to eliminate S we get the very useful relation

$$V_0\rho_0\left(U_1 - U_0\right)^2 = (P_1 - P_0)(V_0 - V_1). \qquad \text{(I.2b)}$$

3. Conservation of Energy

Conservation of energy requires that the net work on the mass equal the change in kinetic and internal energy.

$$(P_1U_1 - P_0U_0)t = L\rho_0\left[\tfrac{1}{2}\left(U_1{}^2 - U_0{}^2\right) + E_1 - E_0\right],$$

where E_1 and E_0 represent internal energy per unit mass.

$$P_1U_1 - P_0U_0 = S\rho_0\left[\tfrac{1}{2}(U_1 + U_0)(U_1 - U_0)\right] + S\rho_0[E_1 - E_0]$$

$$= \frac{(P_1 - P_0)}{U_1 - U_0}\left[\tfrac{1}{2}(U_1 + U_0)(U_1 - U_0)\right]$$

$$+ \left(\frac{U_1 - U_0}{V_0 - V_1}\right)(E_1 - E_0)\rho_0V_0,$$

from equations (I.1) and (I.2).

$$P_1U_1 - P_0U_0 = \tfrac{1}{2}(P_1 - P_0)(U_1 + U_0) + \frac{U_1 - U_0}{V_0 - V_1}(E_1 - E_0)\rho_0V_0,$$

$$\left(\frac{P_1 + P_0}{2}\right)(U_1 - U_0) = \frac{U_1 - U_0}{V_0 - V_1}(E_1 - E_0)\rho_0V_0,$$

$$(E_1 - E_0)\rho_0V_0 = \frac{P_1 + P_0}{2}(V_0 - V_1),$$

or, including ρ_0V_0 in the energy units

$$E_1 - E_0 = \frac{P_1 + P_0}{2}(V_0 - V_1), \qquad \text{(I.3)}$$

where $E = \rho_0V_0E$ (or E is in units of the reference density).

Equations (I.1), (I.2), and (I.3) are the Hugoniot relations expressing conservation of mass, momentum, and energy.

4. Eliminating Energy Dependence

For a given equation of state $P = P(V, E)$ the energy dependence may be eliminated by the third Hugoniot equation. This gives the Hugoniot or pressure-volume points that may be reached by a shock starting from a reference state (Fig. 2).

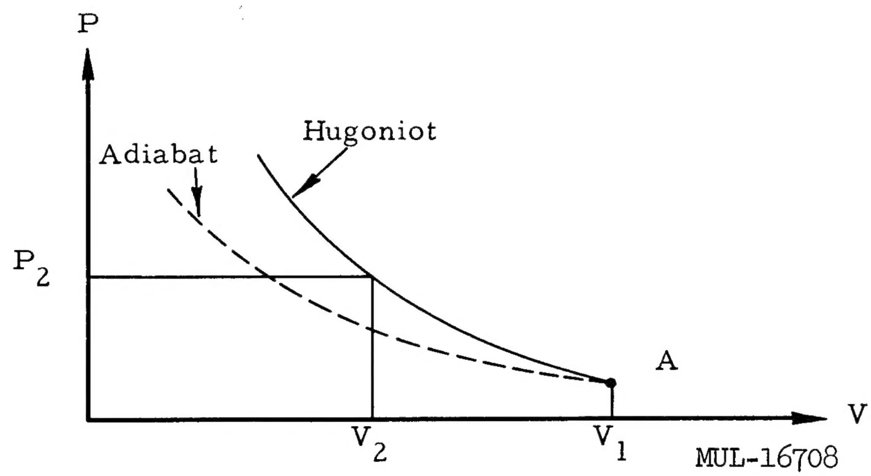

Fig. 2. P vs. V for a given equation of state.

For small $V_2 - V_1 = dv$, we have $\frac{1}{2}(P_1 + P_2) \approx P_1$ and $E_2 - E_1 = dE$ $= P\,dv$, or the adiabat and the Hugoniot coincide.

II. APPLYING THE HUGONIOT RELATIONS TO A PERFECT GAS

1. Propagating a Uniform Shock into a Perfect Gas at Rest

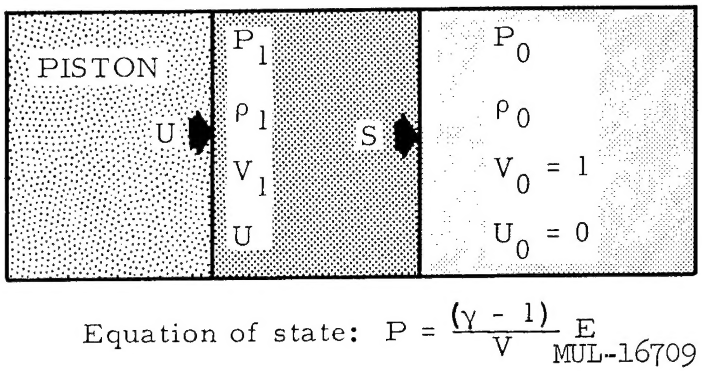

Fig. 3. Propagation of a uniform shock into a perfect gas at rest.

Given a column of gas at rest, at one end a piston is suddenly given the velocity U which is maintained constant. A shock S travels down the column changing the gas from the state subscript zero to subscript one. We wish to find the new state variables and the shock speed S.

$$P_1 - P_0 = \rho_0 US \quad \text{(Second Hugoniot equation),} \quad (II.1)$$

$$U = S(1 - V_1) \quad \text{(First Hugoniot equation),} \quad (II.2)$$

$$P_1 = P_0 \left[\frac{(\gamma + 1) - V_1(\gamma - 1)}{V_1(\gamma + 1) - (\gamma - 1)} \right] \quad \text{(Third Hugoniot equation and the equation of state),} \quad (II.3)$$

or

$$P_1 = P_0 \left[\frac{2 + (1 - V_1)(\gamma - 1)}{2 - (1 - V_1)(\gamma + 1)} \right]$$

Eliminating S from equations (II.1) and (II.2) gives

$$P_1 = P_0 + \frac{\rho_0 U^2}{(1 - V_1)},$$

and substituting in equation (II.3), we get

$$\left(1 - V_1\right)^2 + \frac{(\gamma + 1)\rho_0 U^2}{2\gamma P_0} (1 - V_1) - \frac{\rho_0 U^2}{\gamma P_0} = 0. \qquad \text{(II.4)}$$

Replacing $(1 - V_1)$ by U/S from equation (II.2) we get

$$S^2 - \frac{\gamma + 1}{2} US - \frac{\gamma P_0}{\rho_0} = 0. \qquad \text{(II.5)}$$

This equation gives the shock speed S when the piston velocity and the state ahead of the shock are known. The roots of the equation are always real, one positive and the other negative corresponding to whether the gas is to the left or right of the piston that is compressing it.

The positive part of equation (II.5) is

$$S = \frac{\gamma + 1}{4} U + \sqrt{\left(\frac{\gamma + 1}{4}\right)^2 U^2 + \frac{\gamma P_0}{\rho_0}}. \qquad \text{(II.6)}$$

The speed of sound ahead of the shock is $C_0 = \sqrt{\gamma P_0/\rho_0}$. Hence

$$S = \frac{\gamma + 1}{4} U + \sqrt{\left(\frac{\gamma + 1}{4}\right)^2 U^2 + C_0^{\,2}}.$$

The shock speed S increases with U. When U is zero $S = C_0$. For large U, where C_0/U is negligible, $S = \frac{\gamma + 1}{2} U$.

From equation (II.1) we have $S = \frac{P_1 - P_0}{\rho_0 U}$. Replacing S by this value in equation (II.5) we get an expression for the pressure P_1.

$$\left(P_1 - P_0\right)^2 - \frac{\gamma + 1}{2} \rho_0 U^2 (P_1 - P_0) - \gamma \rho_0 P_0 U^2 = 0. \qquad \text{(II.7)}$$

Solving equation (II.7) for the positive root gives

$$P_1 = P_0 + \frac{\gamma + 1}{4} \rho_0 U^2 + \rho_0 U \sqrt{\left(\frac{\gamma + 1}{4}\right)^2 U^2 + C_0^{\,2}}. \qquad \text{(II.8)}$$

When U is large enough that C_0/U is negligible,

$$P_1 - P_0 = \left(\frac{\gamma + 1}{2}\right) \rho_0 U^2.$$

The relative volume V_1 behind the shock is obtained by eliminating S from equations (II.2) and (II.6).

$$V_1 = 1 - \frac{U}{\left(\frac{\gamma + 1}{4}\right)U + \sqrt{\left(\frac{\gamma + 1}{4}\right)^2 U^2 + C_0^{\ 2}}} \ . \qquad \text{(II.9)}$$

When C_0/U is negligible,

$$V_1 = \frac{\gamma - 1}{\gamma + 1} \ ,$$

which is the minimum relative volume that a single shock can produce.

2. Graphical Representation

To get a clearer view of these relations we shall represent the Hugoniot and adiabat graphically. Taking as coordinates P_1/P_0 and V_1 (remembering that V_1 is the relative volume behind the shock where the relative volume ahead of the shock is 1) and using equations (II.3) for the Hugoniot we get the curves shown in Fig. 4.

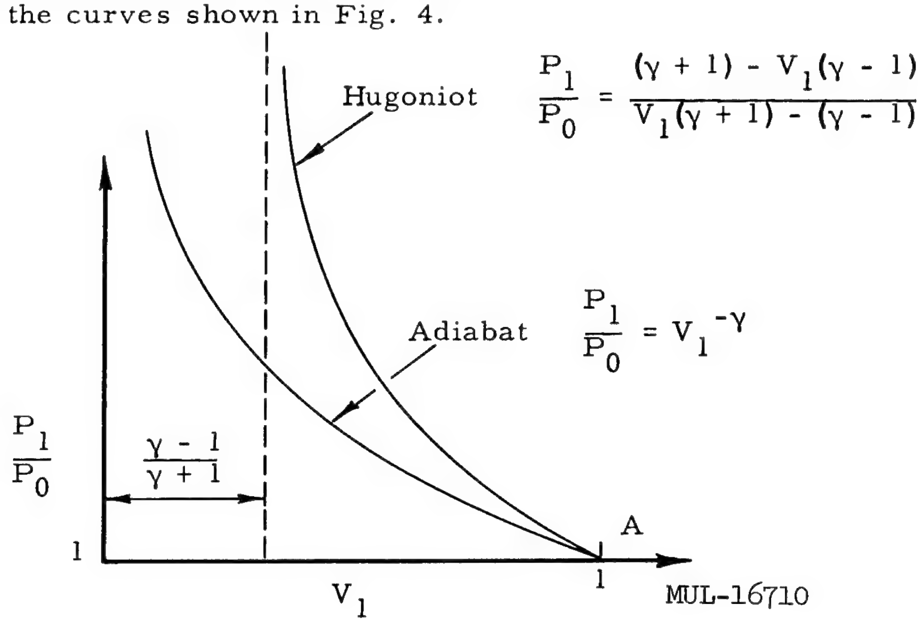

Fig. 4. Hugoniot and adiabat starting from same point.

The Hugoniot starting from point A has the asymptote $\frac{\gamma - 1}{\gamma + 1}$ while the adiabat has the ordinate as an asymptote. By differentiating the Hugoniot and the adiabat two times with respect to V_1, we get

$$\left[\frac{d(P_1/P_0)}{dV_1}\right]_{A\,(\text{Hugoniot})} = \left[\frac{d(P_1/P_0)}{dV_1}\right]_{A\,(\text{adiabat})} = -\gamma,$$

$$\left[\frac{d^2(P_1/P_0)}{dV_1^{\,2}}\right]_{A\,(\text{Hugoniot})} = \left[\frac{d^2P_1/P_0}{dV_1^{\,2}}\right]_{A\,(\text{adiabat})} = \gamma(\gamma + 1),$$

which shows that the Hugoniot and adiabat have at point A the same tangent and curvature, as was pointed out in section I.

3. Reflection of a Uniform Shock

In the preceding analysis we have considered the column of gas to be infinite in length. Now, we shall suppose that it is terminated by a "stone wall" where the velocity is always zero.

When the shock S from the piston, equation (II.6), reaches the stone wall a reflection is produced. That is, a new shock S_1 is formed which travels back toward the piston and changes the gas velocity from U_1 to that of the stone wall or zero. This shock reaches the piston and a new shock S_2, analogous to S, is formed, etc.

We wish to find the values of P_n and V_n behind a shock S, where for odd _n_ the shock is traveling from the piston toward the stone wall and for even _n_ the shock is traveling from the stone wall toward the piston.

$$S_n = \frac{(U_n - U_{n-1})V_{n-1}}{V_{n-1} - V_n}, \qquad \text{(II.10)}$$

$$P_n - P_{n-1} = \rho_{n-1}S_n(U_n - U_{n-1}). \qquad \text{(II.11)}$$

[Note that equations (II.10) and (II.11) are generalized versions of equations (I.1) and (I.2), respectively.] The quantity $(U_n - U_{n-1})$ alternates from (+) the piston velocity to (-) the piston velocity U. For a shock process, $P_n > P_{n-1}$, so S changes sign corresponding to its direction. Eliminating S from equations (II.10) and (II.11) and taking account of the fact that $\rho_{n-1} = \rho_0/V_{n-1}$, we get

$$V_n = V_{n-1} - \frac{\rho_0 U^2}{P_n - P_{n-1}}, \qquad \text{(II.12)}$$

$$P_n = P_{n-1} + \frac{\gamma + 1}{4} \rho_{n-1} U^2 + \rho_{n-1} U \sqrt{\left(\frac{\gamma + 1}{4}\right)^2 U^2 + C_{n-1}{}^2}. \qquad \text{(II.13)}$$

[Equation (II.13) is derived from equation (II.8).]

The solution of the problem is complete since all the quantities of index n can be gotten from the quantities of index n - 1.

4. Application

We wish to apply equations (II.12) and (II.13) to get the conditions behind the first reflected shock from the stone wall. Considering the initial state of the gas to be $V_0 = 1$, $P_0 = 0$, and density ρ_0, we get from equations (II.12) and (II.13) the state behind the incoming shock:

$$V_1 = \frac{\gamma - 1}{\gamma + 1},$$

$$P_1 = \frac{\gamma + 1}{2} \rho_0 U^2,$$

$$\rho_1 = \frac{\gamma + 1}{\gamma - 1} \rho_0.$$

Substituting these values back in equations (II.12) and (II.13), we get the state behind the shock reflected from the wall:

$$P_2 = \left(\frac{3\gamma - 1}{\gamma - 1}\right) P_1,$$

$$V_2 = \frac{\gamma - 1}{\gamma} V_1.$$

III. GRAPHICAL METHODS OF SOLUTION

Problem 1

A plate P_1 originally traveling to the right with velocity U_0 strikes a target which is at rest (see Fig. 5). After collision, the velocity U_0 of the plate is U_3. A shock S_3 is transmitted to the right into the target and a shock S_2 is transmitted to the left into the plate. The shock S_3 changes the

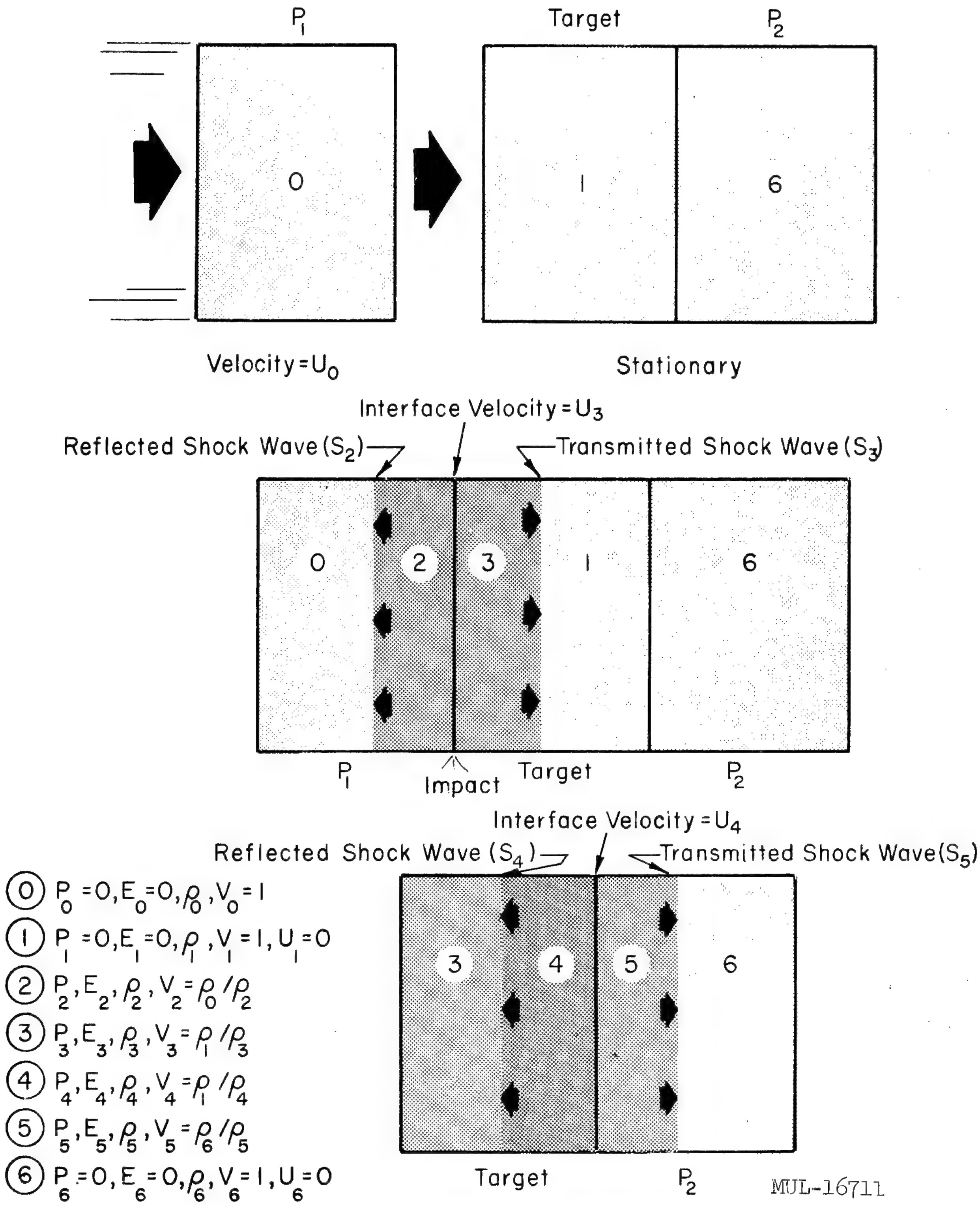

Fig. 5. Plate P_1 traveling to the right at velocity U_0 strikes stationary target, creating shocks S_2 and S_3 in plate and target as shown. Shock S_3 in target travels to interface with second plate, P_2, and creates transmitted and reflected shocks S_5 and S_4, respectively.

target material from state P_1, E_1, ρ_1 to state P_3, E_3, ρ_3. Similarly the shock S_2 into the plate changes the plate material from state P_0, E_0, ρ_0 to state P_2, E_2, ρ_2. The pressures at the interface, P_2 and P_3, will be equal. [See equations (III.1).]

Equations (III.1)

For the plate P_1	For the target
From Hugoniot mass and momentum conservation equations:	
$\rho_0\left(U_3 - U_0\right)^2 = (P_2 - P_0)(V_0 - V_2)$	$\rho_1\left(U_3 - U_1\right)^2 = (P_3 - P_1)(V_1 - V_3)$
Equation of state:	
$P_2 = P_{plate}(V_2, E_2)$	$P_3 = P_{target}(V_3, E_3)$
Third Hugoniot equation:	
$E_2 - E_0 = \dfrac{P_2 + P_0}{2}(V_0 - V_2)$	$E_3 - E_1 = \dfrac{P_3 + P_1}{2}(V_1 - V_3)$

The third Hugoniot equation can be used to eliminate E from the equation of state to give the "Hugoniot equation" for the material where P is a function of V alone. In this problem we will consider the pressure and energy for the plate and target to be zero before the collision.

Rewriting the equations (III.1) with Hugoniots for the equations of state we have:

Equations (III.2)

For the plate	For the target
$\rho_0\left(U_3 - U_0\right)^2 = P_2(V_0 - V_2)$	$\rho_1\left(U_3 - U_1\right)^2 = P_2(V_1 - V_3)$
$P_2 = H_{plate}(V_2)$	$P_3 = H_{target}(V_3)$

We are given the conditions before the collision: U_0, V_0, and ρ_0 for the plate, and U_1, V_1, and ρ_1 for the target. Since $P_2 = P_3$ there are four unknowns — P_2, V_2, V_3, and U_3 — and we have four equations. The four equations are not readily solved algebraically, so a graphical method will be used. Curves of P vs ΔU are made for each material by assigning values to V_2 and V_3, respectively, in the equations (III.2) and calculating the resulting change in velocity ΔU. These curves (Fig. 6) then represent all of the possible pressure-velocity states each material could have as a result of a strong shock starting from an initial state $P = 0$ and $E = 0$.

The change in velocity, ΔU, may be positive or negative: the sign is negative if the material is slowed down from a given velocity by hitting an object and the sign is positive if the material is speeded up by an object hitting it. The problem of a graphical solution of a plate hitting a target is to get the two P-vs-ΔU curves into the same coordinate system.

If we consider the coordinate frame of the target where in this case $U_1 = 0$ and U_3 is an increasing velocity to the right, then in this frame the plate has $P = 0$ when $U_3 = U_0$ and the pressure will increase as U_3 decreases from this point. Hence the solution to the problem is obtained by superimposing the curves as shown in Fig. 7.

The volumes V_2 and V_3 are found by substituting the values of P and U_3 at the intersection of the two curves into equations (III.2).

In practice the Hugoniot curves for various materials are made up in advance in the form of P-vs-ΔU curves where the initial state is considered to be $U = 0$, $P = 0$, $V = 1$, and $E = 0$. Then the left side of the P-vs-ΔU curve is obtained by "flipping" the curve about the ordinate as was done above.

From Fig. 7 it can be seen that if the target and plate were of the same material, the value of U_3 at the interface would be 1/2 the incoming velocity U_0. If the target material were infinitely stiff as a "stone wall," where $U_3 = 0$ for all P, the pressure at collision would be the value of P for the incoming velocity U_0. If the material were infinitely soft or a "void," with $P = 0$ for all U_3, then pressure at collision would be zero and U_3 would remain equal to U_0.

Problem 2

Consider a second plate to be at rest to the right of the target in problem 1 (see Fig. 5). The shock S_3 traveling through the target meets the second

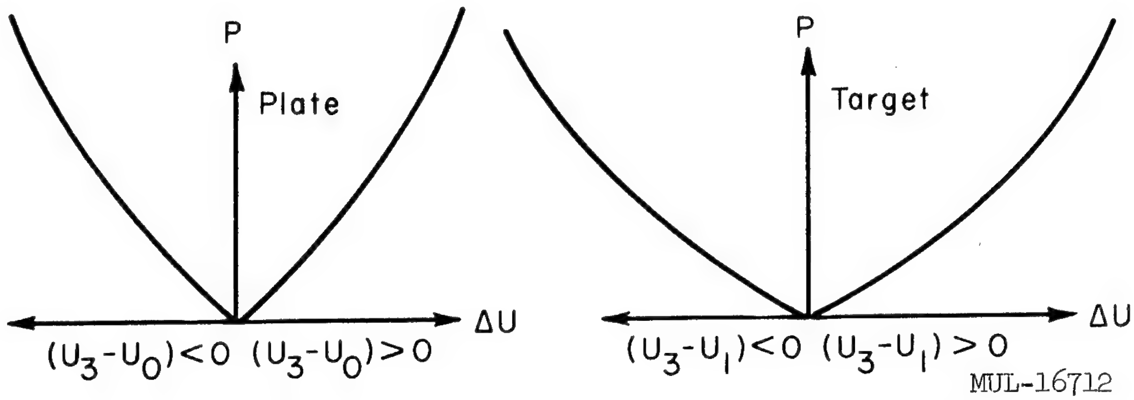

Fig. 6. Curves of P vs ΔU for plate and target.

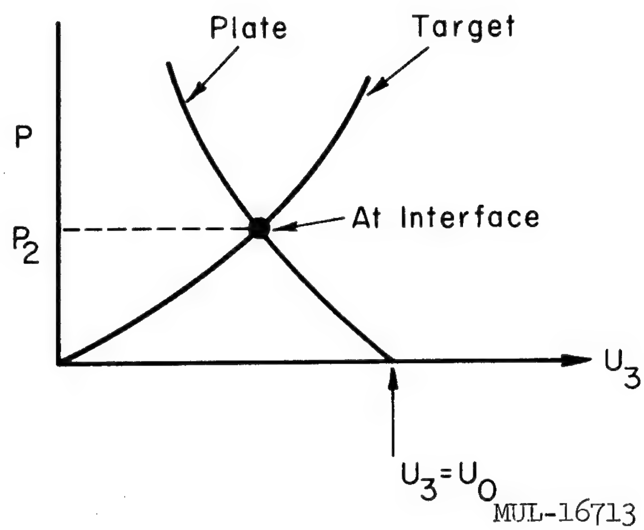

Fig. 7. Superimposing plate and target curves from Fig. 6 to solve for P and U at interface.

plate, P_2, and transmits a shock S_5 into it, changing the state of material from subscript 6 to subscript 5 as shown. The reflected signal S_4 transmitted into the target may be a shock or a rarefaction; however, the signal in the plate will always be a shock. The velocity at the interface is U_4, and again the pressures at the interface, P_4 and P_5, are equal.

First we shall assume S_4 to be a shock. Writing the Hugoniot equations as before, we have equations (III.3)

Equations (III.3)

For the target	For plate 2
$\rho_1\left(U_4 - U_3\right)^2 = (P_4 - P_3)(V_3 - V_4)$	$\rho_6\left(U_4 - U_6\right)^2 = (P_5 - P_6)(V_6 - V_5)$
$P_4 = H_{target}(V_4)$	$P_5 = H_{plate\ 2}(V_5)$

We are given the conditions just as the shock reaches the interface: U_3, V_3, and ρ_1 for the target, and U_6, V_6, and ρ_6 for plate 2. Since $P_4 = P_5$, again there are four unknowns — P_4, V_4, V_5, and U_4 — and four equations. The equations are solved graphically as before, except here the Hugoniot equation for the target is not the same since it is the locus of pressure-volume points starting for $E_3 \neq 0$ and $P_3 \neq 0$. It is obtained in the same manner by substituting the third Hugoniot equation into the equation of state $\left[\text{i.e., substitute } E_4 - E_3 = \left(\frac{P_4 + P_3}{2}\right)(V_3 - V_4) \text{ in the equation of state } P = P_{target}(V_4, E_4) \text{ so as to eliminate } E_4\right]$. This Hugoniot compares with the Hugoniot from $E = 0$, $P = 0$, $V = 1$ as shown schematically in Fig. 8. The target Hugoniot starting from V_3 and P_3 and the plate Hugoniot are used to construct P-vs-ΔU curves by using equations (III.3) in the same way as was done before.

The graphical solution consists of getting the target P-vs-ΔU curve into the coordinate frame of the plate as shown in Fig. 9.

The volumes V_4 and V_5 are found by substituting the values of P_4 and U_4 at the intersection of the two curves in the equations (III.3).

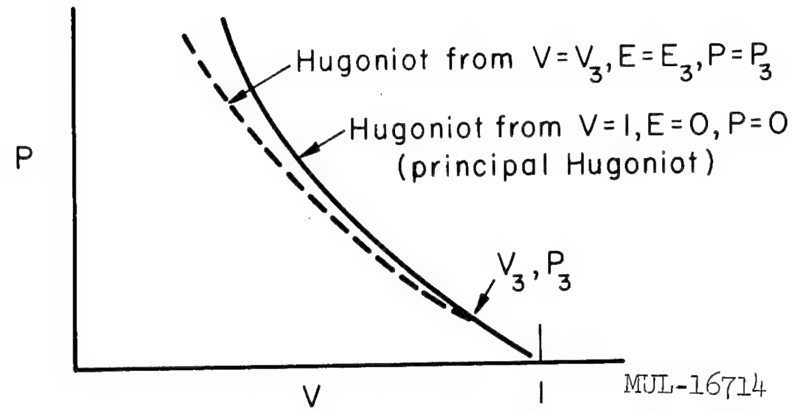

Fig. 8. Hugoniot from point (V_3, P_3) on principal Hugoniot.

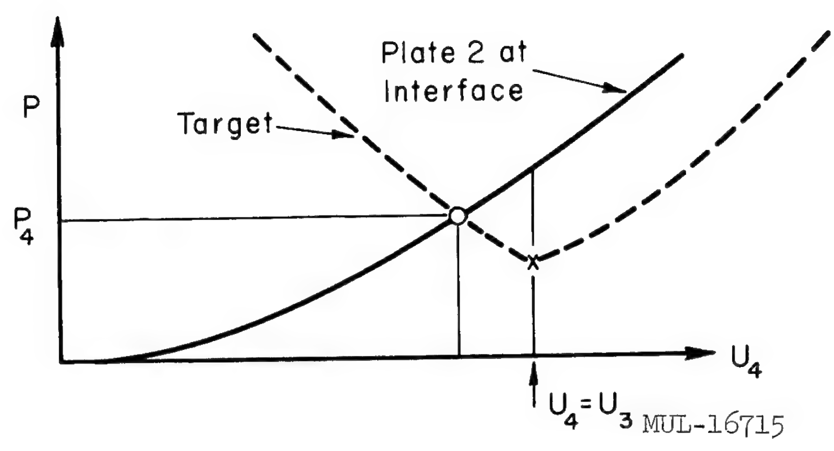

Fig. 9. Graphical solution for P and U at interface of target and plate 2.

For most solid metals the energy dependence in the equations of state is small so that over limited ranges the Hugoniot starting from V_3, P_3, E_3 in Fig. 8 turns out to be very close to the principal Hugoniot. Also, over limited ranges the adiabat expanding from a point on the Hugoniot will lie close to the Hugoniot. These facts greatly simplify and increase the range of problems that can be solved in this manner. (See Appendix for an analysis on a typical equation of state for a metal.)

Because of the small energy dependence it will not be necessary in problem 2 to construct the P-vs-ΔU curve starting with the conditions $V = V_3$, $P = P_3$, $E = E_3$, and $U = U_3$; instead we may use the P-vs-ΔU curve that was obtained from the state $P = 0$, $E = 0$, $V = 1$, and $V = 0$. We still have the problem of getting into the right coordinate frame.

We want to match the velocities so that the target is moving at velocity U_3 in the coordinate frame of the plate. This is accomplished by flipping the target P-vs-U curve about U_3 as shown in Fig. 10.

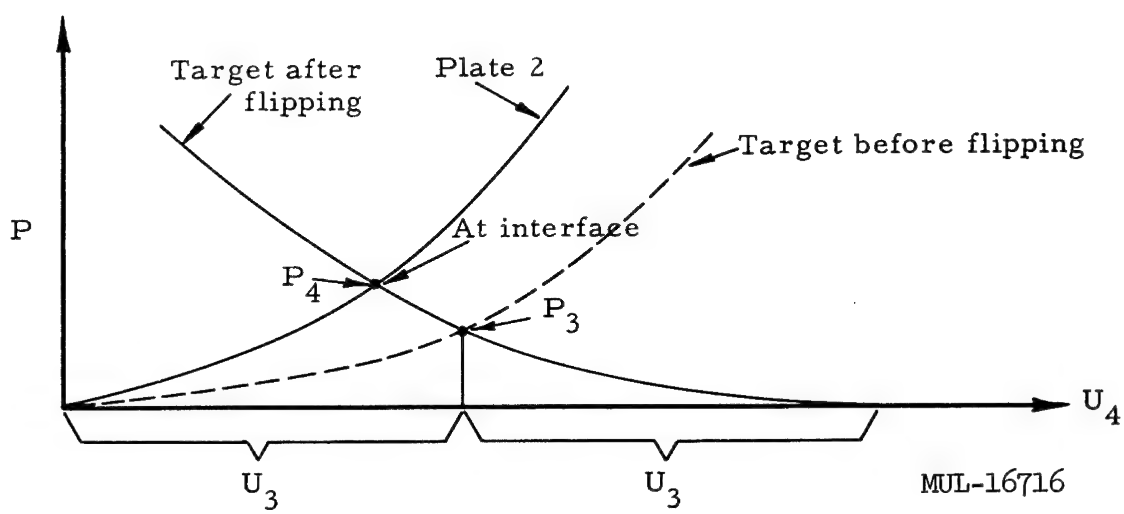

Fig. 10. Graphical solution for P and U at interface, using the principal Hugoniots for plate and target.

From Fig. 10 we see that the target pressure increased from P_3 to P_4, indicating a shock. If plate 2 had been the same material as the target the P-vs-ΔU curve would have been the same as the target curve before flipping and no change in pressures would have occurred at the interface. If the plate 2 curve had intersected the target curve to the right of P_3 the pressure would have dropped in the target, indicating a rarefaction. If plate 2 had been a void (P = 0 for all U) the pressure P_3 would have gone to zero and the velocity from U_3 to $2U_3$. Thus the front surface velocity is equal to twice the particle velocity for a shock traversing the material and reaching a free surface.

In equation-of-state measurements the free surface velocity is measured for an unknown material. From this the particle velocity is known, and if the shock transit time is measured the pressure can be obtained by the Hugoniot relation $P = \rho_0 US$. The other Hugoniot relations give the volume and energy. A series of P, V, E points can be obtained in this way by repeating the experiment with various input velocities. These points do not allow one to immediately write the pressure, volume, energy relations or the surface that represents all the states of the material since we only have the experimental points of a line on this surface. To write the equation of state, assumptions are made concerning the energy dependence and a P, V, E relation is written satisfying the experimental Hugoniot points.

Problem 3

Consider a length of high explosive (H.E.) that has been detonated, where the detonation front has just reached the surface of an adjoining plate at rest. A shock S_1 is transmitted into the plate. The problem is to find the pressure, density, and internal energy behind the shock. (See Fig. 11.)

The three conservation laws must still hold. The problem is similar to problem 2, but here the detonation front replaces the shock S_3. The Hugoniot curve for the H.E. is constructed starting from the conditions at the detonation front, called the Chapman-Jouguet point (C-J), and the H.E. equation of state, all considered to be known in advance. The procedure is identical to problem 2, with the subscript 3 being replaced by the values at the C-J point. The P-vs-U curves for both materials are superimposed in Fig. 12 (just as was done in Fig. 9), and the intersection gives the values of

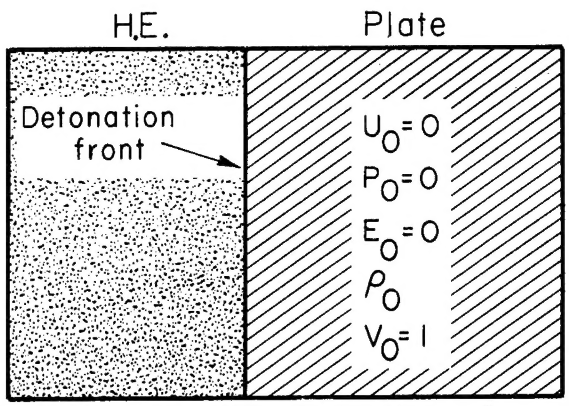

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Fig. 11. Detonation front in high explosive (H. E.) just as it reaches surface of adjoining plate.

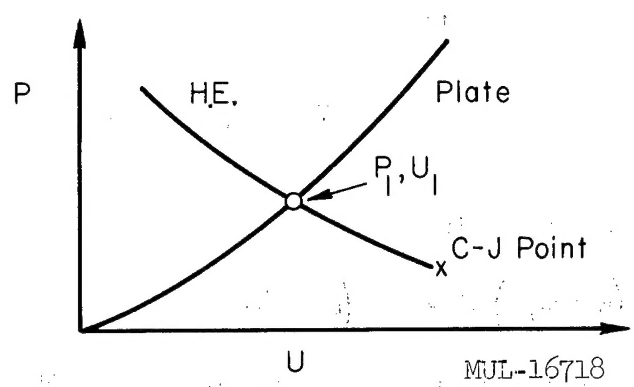

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Fig. 12. Graphical solution to find P and U at the interface between H. E. and plate of Fig. 11.

$$P = \underline{a\mu + \left(b + \frac{ac}{2}\right)\mu^2 + \frac{c}{2}\left(b + \frac{ac}{2}\right)\mu^3} - \frac{c}{2}\mu\left[a\frac{\delta}{V} + \left(b + \frac{ac}{2}\right)\mu\frac{\delta}{V} - \frac{c}{2}a\frac{\delta^2}{V^2}\right]$$
$$+ \frac{c}{2}\mu\left[a\mu_1 + \left(b + \frac{ac}{2}\right)\mu_1^2 + \frac{1}{2}c\left(b + \frac{ac}{2}\right)\mu_1^3 + \ldots\right]. \qquad \text{(A.2a)}$$

The underlined terms in equation (A.2a) are the same as the principal Hugoniot. Since $\frac{\delta}{V} \approx \mu_1$ and $\underline{c}$ is usually between 1 and 2, equation (A.2a) is very nearly equal to the principal Hugoniot except for high order terms.

APPENDIX B

To show that the front surface velocity equals twice the particle velocity ($U_{fs} = 2U_p$) for a material with a Hooke's law equations of state, $P = a(\rho/\rho_0 - 1)$.

$$c^2 = \frac{\partial P}{\partial \rho} = \frac{a}{\rho_0}$$

where

a = constant,

ρ_0 = reference density,

c = sound speed.

The Riemann invariant, σ, for the hydrodynamic equations of motion is $\sigma = \int c\frac{d\rho}{\rho}$. (See R. Courant and K. O. Fredricks, Supersonic Flow and Shock Waves, Interscience, New York, 1948.)

Since c is a constant for this equation of state we have

$$\sigma = c\int\frac{d\rho}{\rho} = c\ \ln\frac{\rho}{\rho_0} = c\ \ln\eta,$$

where $\eta = \rho/\rho_0$.

For a shock traveling through undisturbed material we have

$$\rho_0 U_p^2 = P(1 - V).$$

We want to find the front surface velocity when the shock reaches it. Taking the +c characteristics we have

$$U_p + \sigma_p = U_{fs} + \sigma_{fs}$$

at the front surface $\sigma = 0$. Hence

$$U_{fs} = U_p + \sigma_p .$$

Substituting the equation of state into the relation $\rho_0 U_p{}^2 = P(1 - V)$, we get

$$\rho_0 U_p{}^2 = a(\eta - 1)\left(1 - \frac{1}{\eta}\right)$$

$$= \frac{a}{\rho_0} \frac{(\eta - 1)^2}{\eta}, \quad \text{where} \quad \eta = \frac{1}{V} = \frac{\rho}{\rho_0},$$

$$U_p = c\frac{\eta - 1}{\sqrt{\eta}},$$

$$c = U_p \frac{\sqrt{\eta}}{\eta - 1},$$

$$\sigma_p \doteq c \ln \eta = U_p \frac{\sqrt{\eta}}{\eta - 1} \ln \eta$$

$$= U_p, \quad \text{since} \quad \frac{\sqrt{\eta}}{\eta - 1} \ln \eta \approx 1.$$

From $$U_{fs} = U_p + \sigma_p,$$

$$\underline{U_{fs} = 2U_p}.$$

In the above derivation the equation of state was assumed to be linear in η. A more critical analysis, using a nonlinear equation of state, will show the preceding result is valid over a large range of pressure and compression. Consequently the assumption of using the principal Hugoniot in solving shock interaction problems is reasonable.

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